12/10/15.

$A \subseteq [-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\} = \overline{\mathbb{R}}$

$\inf \sup \{x_n : n \geq k\} = \limsup x_n = \overline{\lim}\, x_n$

$\sup \inf \{x_n : n \geq k\} = \liminf x_n = \underline{\lim}\, x_n$

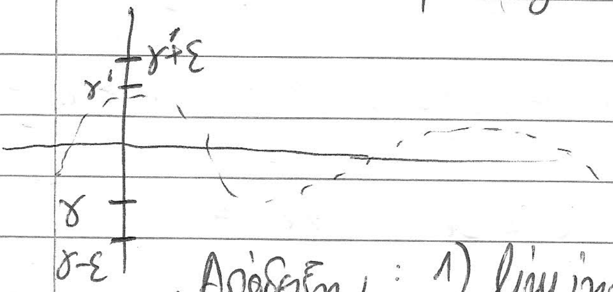
$$\boxed{\begin{aligned} \limsup(-x_n) &= -\liminf x_n \\ \liminf(-x_n) &= -\limsup x_n. \end{aligned}}$$

$A \subseteq \overline{\mathbb{R}} \Rightarrow -A = \{-x, x \in A\}$

$$\boxed{\begin{aligned} \sup(-A) &= -\inf A \\ \inf(-A) &= -\sup A \end{aligned}} \qquad \boxed{\liminf x_n \leq \limsup x_n}$$

1) $\gamma \in \mathbb{R}$: $\liminf x_n \geq \gamma \iff (\forall \varepsilon > 0)(\exists n_0) : n \geq n_0 \Rightarrow x_n \geq \gamma - \varepsilon$

$\limsup x_n \leq \gamma' \iff (\forall \varepsilon > 0)(\exists n_1) : n \geq n_1 \Rightarrow x_n \leq \gamma + \varepsilon$



Απόδειξη : 1) $\liminf x_n \geq \gamma$. Έστω $\varepsilon > 0$.
Τότε $\liminf x_n \geq \gamma > \gamma - \varepsilon \Rightarrow \boxed{\liminf x_n > \gamma - \varepsilon}$

δηλ. $\sup\limits_k \inf\{x_n : n \geq k\} > \gamma - \varepsilon \overset{\text{άρα}}{\Rightarrow} \exists k : \inf\{x_n : n \geq k\} > \gamma - \varepsilon \Rightarrow$

$\Rightarrow x_n > \gamma - \varepsilon$, $\forall n \geq k$. άρα για $k = n_0$ έχω το ζητούμενο.

$\limsup x_n \leq \gamma' \Rightarrow -\limsup x_n \geq -\gamma' = \gamma \Rightarrow \liminf(-x_n) \geq \gamma \iff$
$\iff (\forall \varepsilon > 0)(\exists n_0) : -x_n \geq \gamma - \varepsilon \Rightarrow x_n \leq -\gamma + \varepsilon = \gamma' + \varepsilon.$

2) Αν $\limsup x_n = \liminf x_n := L \in \overline{\mathbb{R}} \Leftrightarrow \exists \lim x_n = L$

$(\forall \varepsilon > 0)(\exists n_0)$
$n \geq n_0 \Rightarrow$
$|x_n - L| < \varepsilon$

α) $L \in \mathbb{R}$: ~~[illegible]~~ $\limsup x_n = L$.
Έστω $\varepsilon > 0$. Αφού $\limsup x_n = L \Rightarrow \boxed{\limsup x_n \leq L} \Rightarrow$

$\Rightarrow (\exists n_1) : n \geq n_1 \Rightarrow \boxed{x_n \leq L + \varepsilon}$

$\liminf x_n = L \Rightarrow \boxed{\liminf x_n \geq L} \Rightarrow (\exists n_0) : n \geq n_0 \Rightarrow \boxed{x_n \geq L - \varepsilon}$

Τότε $\bar{n} = \max\{n_0, n_1\}$ και άρα για $n \geq \bar{n} \Rightarrow \begin{cases} n \geq n_0 \\ n \geq n_1 \end{cases} \Rightarrow$

$\Rightarrow \left.\begin{array}{l} x_n \geq L - \varepsilon \\ x_n \leq L + \varepsilon \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_n - L \geq -\varepsilon \\ x_n - L \leq \varepsilon \end{array}\right\} \Rightarrow |x_n - L| \leq \varepsilon$ (δηλ $\exists \lim x_n$).

β) $L = +\infty$
$\liminf(x_n) = +\infty$ δηλ $(\forall \varepsilon > 0) : \liminf(x_n) > \frac{1}{\varepsilon} + \varepsilon \Rightarrow$

$\Rightarrow \exists n_0 : n \geq n_0 \Rightarrow x_n \geq \gamma - \varepsilon = \frac{1}{\varepsilon} \Rightarrow \boxed{\lim x_n = +\infty}$

Όμως $\liminf \leq \limsup(x_n) \Rightarrow \limsup x_n = +\infty$

3) $(\exists x_{k_n}) : x_{k_n} \longrightarrow \limsup(x_n) = L \Rightarrow$

$\Rightarrow \left.\begin{array}{l} \exists n_1 : n \geq n_1 \Rightarrow x_n \leq L + \varepsilon \\ n_2 = \max\{n_0, n_1\}. \end{array}\right\} \Rightarrow$

$L+\varepsilon \quad L \quad L-\varepsilon$

$(\forall \varepsilon > 0) : x_n \leq L - \varepsilon$
ή $\quad \forall n \geq n_0$
$x_n \geq L + \varepsilon$

$\Rightarrow x_n \leq L - \varepsilon,\ n \geq n_2 \Rightarrow \sup\{x_n : n \geq n_2\} \leq L - \varepsilon$
και άρα $\underline{\inf \sup\{x_n : n \geq n_2\}} \leq L - \varepsilon$
$\boxed{\limsup(x_n) \leq L - \varepsilon}$ <u>ΑΤΟΠΟ</u>

διότι $\limsup(x_n) = L \leq L - \varepsilon$ <u>άτοπο</u> !

δηλ $(\exists x_{k_n}) : \boxed{x_{k_n} \longrightarrow L = \limsup(x_n)}$

Θδο $\limsup(x_n) \le \max\{x' : x' \in \overline{\mathbb{R}}$ κ' $\exists x_{k_n} : x_{k_n} \to x'\}$.

Έστω ότι $\limsup(x_n) < k < \underline{k+\varepsilon \max\{\dots\}}$,

δηλ $\inf_{k}\sup_{n\ge\mu}(x_n) < k \Rightarrow \exists \mu : \sup_{n\ge\mu}(x_n) < k \Rightarrow x_n < k, \forall n \ge \mu$.

Όμως $k+\varepsilon < \max\{\dots\}$ $\Rightarrow$

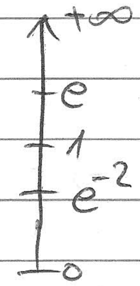
$\Rightarrow$ ισχύει η ισότητα και όχι η γνήσια ανισότητα.

$$x_n = \begin{cases} 2^{-n} \to 0, & n = 5\lambda \\ \left(1+\frac{1}{n}\right)^{n} \to e, & 5\lambda+1 = n \\ 3^{n} \to +\infty, & n = 5\lambda+2 \\ \left(1-\frac{1}{n}\right)^{2n} \to e^{-2}, & n = 5\lambda+3 \\ \cos\left(\frac{1}{n}\right) \to 1, & n = 5\lambda+4 \end{cases}$$

(x_{k_n})

$\{x_n : n\in\mathbb{N}\} = A_0\cup A_1\cup A_2\cup A_3\cup A_4$

Ένα από τα A_i, $i=0,\dots,4$ θα έχει άπειρους όρους.



Θδο $\limsup(a_n+b_n) \le \limsup a_n + \limsup b_n$

Άσκ: Είναι: $\liminf(a_n+b_n) \ge \liminf(a_n) + \liminf(b_n)$

$$\limsup(a_n+b_n) = \inf_k \sup\{a_n+b_n : n\ge k\}$$

$\{a_n+b_n : n\ge k\}$, $a_k+b_k \le \sup\{a_n : n\ge k\} + b_k \le$

$\le \sup\{a_n : n\ge k\} + \sup\{b_n : n\ge k\}$.

$\sup\{a_n+b_n : n\ge k\} \le \sup\{a_n : n\ge k_1\} + \sup\{b_n : n\ge k_2\}$

άρα $\inf(\quad) \le \inf(\quad) + \inf(\quad)$

άρα

$$\limsup(a_n+b_n) \le \limsup(a_n) + \limsup(b_n)$$

Αν $b_n = b$ (σταθερή) τότε $\limsup(a_n + b) \leq \limsup(a_n) + \limsup(b)$

άρα $\boxed{\limsup(a_n+b) \leq \limsup a_n + b}$ (1)

$\limsup((a_n+b) - b) \leq \limsup(a_n+b) - b$
$\limsup(a_n) \leq \limsup(a_n+b) - b$
$\limsup(a_n) + b \leq \limsup(a_n+b)$

άρα η (1) ισχύει με ισότητα.

$a_n, b_n > 0 \Rightarrow \limsup(a_n \cdot b_n) \leq \limsup(a_n) \cdot \limsup(b_n)$

$\Rightarrow \liminf(a_n \cdot b_n) \geq \liminf(a_n) \liminf(b_n)$.

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n\text{-φορές}} = \{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$

$(x_1, \dots, x_n) + (x_1', \dots, x_n') = (x_1 + x_1', \dots, x_n + x_n')$
$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$.
$(\lambda + \mu)\bar{x} = \lambda\bar{x} + \mu\bar{x}$
$\lambda(\bar{x} + \bar{y}) = \lambda\bar{x} + \lambda\bar{y}$
$|(x_1, \dots, x_n)| = \sqrt{x_1^2 + \dots + x_n^2}$

$\sqrt{(x_1+x_1')^2 + \dots + (x_n + x_n')^2} \leq \sqrt{x_1^2 + \dots + x_n^2} + \sqrt{x_1'^2 + \dots + x_n'^2}$
$|\lambda\bar{x}| = |\lambda| \cdot |\bar{x}|$

$|a - b| = d(a, b) = \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2}$
$a = (a_1, \dots, a_n)$
$b = (b_1, \dots, b_n)$